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Inclusion relations among separability criteria

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Abstract

We revisit the application of different separability criteria by recourse to an exhaustive Monte Carlo exploration involving the pertinent state space of pure and mixed states. The corresponding chain of implications of different criteria is in such a way numerically elucidated. We also quantify, for a bipartite system of arbitrary dimension, the proportion of states ρ that can be distilled according to a definite criterion. Our work can be regarded as a complement to the recent review paper by Terhal B (2002 *Theor. Comput. Sci.* **287** 313). Some questions posed there receive an answer here.

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1. Introduction

The development of criteria for entanglement and separability is one aspect of the current research efforts in quantum information theory that is receiving, and certainly deserves, considerable attention [1]. Indeed, much progress has recently been made in consolidating such a cornerstone of the theory of quantum entanglement [1], earlier pointed out by Schrödinger [2]. The relevant state space here is of a high dimensionality, already 15 dimensions in the simplest instance of two-qubit systems. The systematic exploration of these spaces can provide us with valuable insight into some of the extant theoretical questions.

As a matter of fact, important steps have been recently made towards a systematic exploration of the space of arbitrary (pure or mixed) states of composite quantum systems [3–5] in order to determine the typical features exhibited by these states with regards to the phenomenon of quantum entanglement [3–9]. Entanglement is one of the most fundamental and non-classical features exhibited by quantum systems [10], that lies at the basis of some of the most important processes studied by quantum information theory [10–14] such as quantum cryptographic key distribution [15], quantum teleportation [16], superdense coding [17] and quantum computation [18, 19].

It is well known [1] that, for a composite quantum system, a state described by the density matrix ρ is called 'entangled' if it cannot be represented as a mixture of factorizable pure states. Otherwise, the state is called separable. The above definition is physically meaningful because entangled states (unlike separable states) cannot be prepared locally by acting on each subsystem individually [20].

The question of separability has quite interesting echoes in information theory and its associated information measures or entropies. When one deals with a classical composite system described by a suitable probability distribution defined over the concomitant phase space, the entropy of any of its subsystems is always equal or smaller than the entropy characterizing the whole system. This is also the case for separable states of a composite quantum system [21, 22]. In contrast, a subsystem of a quantum system described by an entangled state may have an entropy greater than the entropy of the whole system. Indeed, the von Neumann entropy of either of the subsystems of a bipartite quantum system described (as a whole) by a pure state provides a measure of the amount of entanglement of such state. Thus, a pure state (which has vanishing entropy) is entangled if and only if its subsystems have an entropy larger than that associated with the system as a whole.

Regrettably enough, the situation is more complex when the composite system is described by a mixed state: there are entangled mixed states such that the entropy of the complete system is smaller than the entropy of one of its subsystems. Alas, entangled mixed states such that the entropy of the system as a whole is larger than the entropy of either of its subsystems do exist as well. Consequently, the classical inequalities relating the entropy of the whole system with the entropy of its subsystems provide only necessary, but not sufficient, conditions for quantum separability. There are several entropic (or information) measures that can be used in order to implement these criteria for separability. Considerable attention has been paid, in this regard, to the q-entropies [1, 22–29], which incorporate both Rényi's [30] and Tsallis' [31–33] families of information measures as special instances (both admitting, in turn, Shannon's measure as the particular case associated with the limit $q \to 1$). The reader is referred to the appendix for a brief review on q-entropies.

The early motivation for the studies reported in [22–29] was the development of practical separability criteria for density matrices. The discovery by Peres [34] of the partial transpose criteria, which for two-qubits and qubit–qutrit systems turned out to be both necessary and sufficient, rendered that original motivation somewhat outmoded. In fact, it is not possible to find a necessary and sufficient criterion for separability based solely upon the eigenvalue spectra of the three density matrices ρ_{AB}, $\rho_A = \mathrm{Tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ associated with a composite system $A \oplus B$ [21].

Interesting concepts that revolve around the separability issue have been developed over the years. An interesting account is given in [1]. Among them we find criteria such as the so-called Majorization, reduction and positive partial transpose (PPT) together with the concept of distillability [1]. Quantum entanglement is a fundamental aspect of quantum physics that deserves to be investigated in full detail from all possible points of view. The chain of implications, and the related inclusion relations, among the different separability criteria is certainly a vantage point worthy of detailed scrutiny. It is our purpose here to revisit, with such a goal in mind, the separability question by means of an exhaustive Monte Carlo exploration involving the whole space of pure and mixed states. Such an effort should shed some light on the inclusion issues that interest us here. Concrete numerical evidence will thus be provided on the relations among the separability criteria. We will then be able to quantify, for a bipartite system of arbitrary dimension, the proportion of states ρ that can be distilled according to a definite criterion. This numerical exploration could be viewed as a complement to the review paper by Terhal [1], because some questions posed by her will receive an answer in this work.

The paper is organized as follows. We sketch in section 2 the different separability criteria to be investigated and discuss some mathematical and numerical techniques used in our survey in section 3. Our results are reported in section 4, and some conclusions are drawn in section 5. For the sake of completeness, we include an appendix on q-entropies.

2. Brief sketch of separability criteria

From a historic viewpoint, the first separability criterion is that of Bell (see [1] and references therein). For every pure entangled state there is a Bell inequality that is violated. It is not known, however, whether in the case of many entangled mixed states, violations exist. There does exist a witness for every entangled state though [35]. It was shown by Horodecki *et al* that a density matrix $\rho \equiv \rho_{AB}$ is entangled if and only if there exists an entanglement witness (a Hermitian super-operator $\hat{W} = \hat{W}^{\dagger}$) such that

$$\mathrm{Tr}\, \hat{W}\rho \leqslant 0 \quad \text{while} \quad \mathrm{Tr}\, \hat{W}\rho \geqslant 0 \quad \text{for all separable states.} \tag{1}$$

A special, but quite important LOCC operational separability criterion, necessary but not sufficient, is provided by the positive partial transpose (PPT) one. Let T stand for matrix transposition. The PPT requires that

$$[\hat{1} \otimes \hat{T}](\rho) \geqslant 0. \tag{2}$$

For cases in which the PPT criterion fails, an interesting numerical procedure that involves the explicit construction of the entanglement witnesses [36] has been found to be successful when applied to low-dimensional states.

Another operational criterion is called the *reduction* criterion, that is satisfied, for a given state $\rho \equiv \rho_{AB}$, when both [1]

$$\hat{1} \otimes \rho_B - \rho \geqslant 0 \qquad \rho_A \otimes \hat{1} - \rho \geqslant 0. \tag{3}$$

Intuitively, the distillable entanglement is the maximum asymptotic yield of singleton states that can be obtained, via LOCC, from a given mixed state. Horodecki *et al* [37] demonstrated that any entangled mixed state of two qubits can be distilled to obtain the singleton. This is not true in general. There are entangled mixed states of two qutrits, for instance, that cannot be distilled, so that they are useless for quantum communication. In our scenario an important fact is that all states that violate the reduction criterion are distillable [38].

Entanglement witnesses completely characterize the set of separable states. Alas, they are not usually associated with a simple computational treatment, except in the PPT instance. Thus, in order to decide whether a given state ρ is entangled one needs additional criteria, functional separability ones [1]. One of them associates PPT with the rank of a matrix. Consider two subsystems A, B whose description is made, respectively, in the Hilbert spaces $\mathcal{H}_n$ and $\mathcal{H}_m$. Focus attention now in the density matrix $\rho \equiv \rho_{AB}$ for the associated composite system. If

(1) ρ has PPT, and
(2) its rank $\mathcal{R}$ is such that $\mathcal{R} \leqslant \max(n, m)$,

then, as was proved in [39], ρ is separable. The above-mentioned entropic criteria are also functional separability ones. Still another one is majorization.

Let $\{\lambda_i\}$ be the set of eigenvalues of the matrix ξ_1 and $\{\gamma_i\}$ be the set of eigenvalues of the matrix ξ_2. We assert that the ordered set of eigenvalues $\vec{\lambda}$ of ξ_1 *majorizes* the ordered set of eigenvalues $\vec{\gamma}$ of ξ_2 (and writes $\vec{\lambda} \succ \vec{\gamma}$) when $\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \gamma_i$ for all k. It has been shown [21] that, for all separable states $\rho_{AB} \equiv \rho$,

$$\vec{\lambda}_{\rho_A} \succ \vec{\lambda}_{\rho} \quad \text{and} \quad \vec{\lambda}_{\rho_B} \succ \vec{\lambda}_{\rho}. \tag{4}$$

There is an intimate relation between this majorization criterion and entropic inequalities, as discussed in [1, 22].

3. Separability probabilities: exploring the whole state space

We promised in the introduction to perform a systematic numerical survey of the properties of arbitrary (pure and mixed) states of a given quantum system by recourse to an exhaustive exploration of the concomitant state space $\mathcal{S}$. To such an end it is necessary to introduce an appropriate measure μ on this space. Such a measure is needed to compute volumes within $\mathcal{S}$, as well as to determine what is to be understood by a uniform distribution of states on $\mathcal{S}$. The measure that we are going to adopt here is taken from the work of Zyczkowski *et al* [3, 4]. An arbitrary (pure or mixed) state ρ of a quantum system described by an N-dimensional Hilbert space can always be expressed as the product of three matrices,

$$\rho = UD[\{\lambda_i\}]U^{\dagger}. \tag{5}$$

Here U is an $N \times N$ unitary matrix and $D[\{\lambda_i\}]$ is an $N \times N$ diagonal matrix whose diagonal elements are $\{\lambda_1, \ldots, \lambda_N\}$, with $0 \leqslant \lambda_i \leqslant 1$, and $\sum_i \lambda_i = 1$. The group of unitary matrices $U(N)$ is endowed with a unique, uniform measure: the Haar measure ν [40]. On the other hand, the N-simplex Δ, consisting of all the real N-uples $\{\lambda_1, \ldots, \lambda_N\}$ appearing in (5), is a subset of a $(N-1)$-dimensional hyperplane of $\mathcal{R}^N$. Consequently, the standard normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathcal{R}^{N-1}$ provides a measure for Δ. The aforementioned measures on $U(N)$ and Δ lead then to a measure μ on the set $\mathcal{S}$ of all the states of our quantum system [3, 4, 40], namely,

$$\mu = \nu \times \mathcal{L}_{N-1}. \tag{6}$$

In our numerical computations we randomly generate states according to the measure (6). This measure, as pointed out by Slater [41–44] can be criticized on two grounds: on the one hand, it is not associated with the volume element of any monotonic metric; on the other one, it can be regarded as over-parametrized (and in this sense, arbitrary) because the number of variables it needs to parametrize the convex set of $N \times N$ density matrices is $N^2 + N - 1$ rather than the theoretical minimum number $N^2 - 1$. For our present purpose, however, this criticism loses some relevance because of the fact that the metric (i) does rapidly converge and (ii) provides a simple procedure to *compare the different separability criteria* in the case of bipartite states (which is our goal here).

Indeed, by recourse to standard Fortran subroutines fast convergence is reached with the use of a powerful laptop when computing the volume ratios (the *a priori* probabilities) under consideration. All the pertinent probabilities (for low and high dimensions) are evaluated by sampling $N_s = 10^8$ states ρ, so that the concomitant error, that diminishes as the inverse square root of N_s, is of the order of 10^{-4}. Convergence is always reached, no matter the dimension one is working with. Additionally, for high dimensions $N = N_1 \times N_2$, the error bars, at the scale of our plots, coincide with the size of the 'dot symbols' that we draw.

Although the absolute values of the involved probabilities may indeed depend on the metrics used to generate the space of mixed states, we have chosen to work with a rather useful measure in order to compare a variety of results obtained according to different criteria. Thus, the conclusions that we reach are based on the assumption that the uniform distribution of states of a quantum system is that determined by the measure (6). Note that we are interested here in *inclusion relations*. These will not depend upon the nature of the measure chosen in the computations.

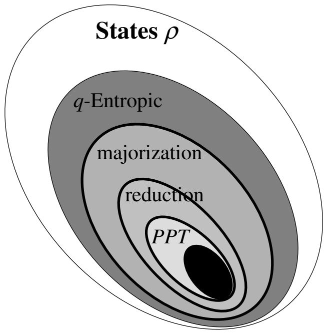

Figure 1. Schematics of the inclusion relations among separability criteria as given by the volume occupied by states ρ for a given dimension N which fulfil them.

4. Survey results

4.1. The overall scenario

An overall picture of the situation we encounter is sketched in figure 1, that is to be compared to figure 3 of [1]. *Note that our numerical exploration allows us to dispense with Terhal's question marks*. This constitutes part of the original content of the present communication.

The set of all mixed states presents an onion-like shape, as conjectured by Terhal [1]. Which among these states are separable? As reviewed above, several criteria are available. We start with the q-entropic one (see the appendix and [45]). By using a definite value of q, namely $q = \infty$, and the sign of the associated, conditional q-entropy, we are able to define a closed sub-region, whose states are presumptively separable. This region has a definite border, that separates it from the sub-region of states entangled according to this criterion. What we see now is that, if we use now *other* separability criteria, the associated sub-regions shrink in a manner prescribed by the particular criterion one employs. The shrinking process ends when one reaches the sub-region defined by the positive partial transpose (PPT) criterion, which is a necessary and sufficient separability condition for 2×2 and 2×3 systems, being only necessary for higher dimensions.

Summing up, the volume of states which are separable according to different criteria diminishes as we use stronger and stronger criteria. There is a first shrinking stage associated with entropic criteria, from its Von Neumann ($q = 1$) size, as q grows, to the limit case $q \to \infty$ [45]. A second stage involves majorization, reduction and finally, positive partial transpose (PPT) [1].

4.2. PPT and reduction

We report now on our state space exploration with regards to the probability of finding a state with positive partial transpose. The results are depicted in figure 2. The solid line corresponds to states with dimension $N = 2 \times N_2$, while the dashed line corresponds to $N = 3 \times N_2$ states. Note how similar are the pertinent values in both cases. The tiny difference between them can be inspected in the inset (a semi-logarithmic plot). To a good approximation, our PPT probabilities decrease exponentially.

Figure 3 deals instead with the probability of finding a state which obeys the strictures of the reduction criterion, for $N = 2 \times N_2$ (solid line) and $N = 3 \times N_2$ (dashed line). As a matter of fact, PPT and reduction coincide for $N = 2 \times N_2$. It is known that if a state satisfies

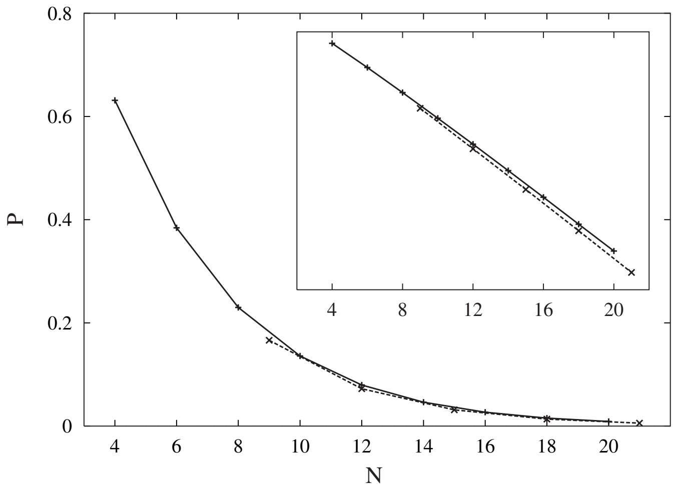

Figure 2. Probability of finding a state with positive partial transpose. The solid line corresponds to states with dimension $N = 2 \times N_2$, while the dashed line corresponds to $N = 3 \times N_2$ states. The difference between these curves can be appreciated in the inset (semi-logarithmic plot). Our probabilities decrease, to a good approximation, in exponential fashion.

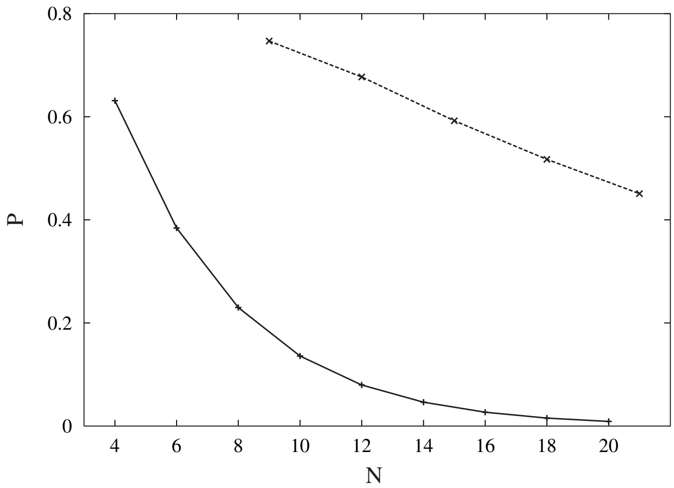

Figure 3. Probability of finding a state fulfilling the reduction criterion for $N = 2 \times N_2$ (solid line) and $N = 3 \times N_2$ (dashed line). For $N = 2 \times N_2$ the probabilities of the reduction and the PPT criteria coincide.

PPT, it automatically verifies the reduction criterion [1]. Here we have demonstrated that, at least in the $N = 2 \times N_2$ instance, the converse is also true. However, in the $N = 3 \times N_2$ case, it is much more likely to encounter a state that verifies reduction than one that verifies PPT.

4.3. Entropic criteria and majorization

We begin with a brief recapitulation of former q-entropic results. The situation encountered in [46] was that the 'best' result within the framework of the 'classical q-entropic inequalities' as a separability criterion was reached using the limit case $q \to \infty$, but considerably less

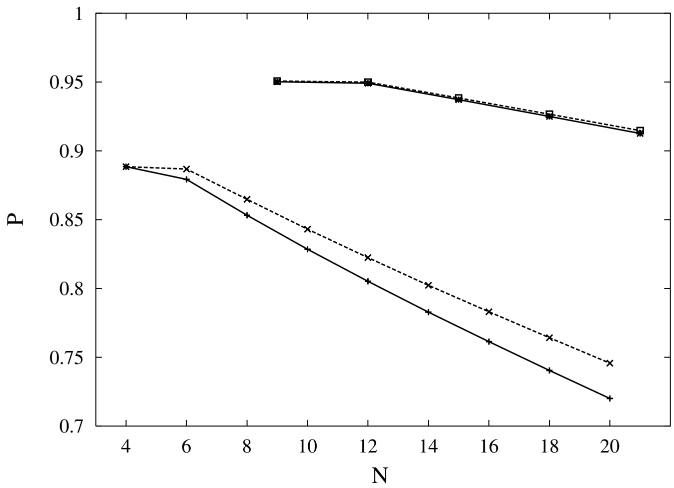

Figure 4. Probability of finding a state whose two relative q-entropies are positive for $q \to \infty$ (dashed curves). The probability that a state be completely majorized by both of their subsystems is represented by the solid line. Bottom: curves correspond to states ρ with $N = 2 \times N_2$. Top: $N = 3 \times N_2$.

attention was paid to other values of q. This was remedied in [45], where the question of q-entropic inequalities for finite q-values was extensively discussed. It was there reconfirmed that the above mentioned limit case does indeed the better job as far as separability questions are concerned [45]. For such a reason, this limit q-value is the only one to be employed below. See the appendix for more details on q-entropies.

In figure 4 we depict the probability of finding a state which, for $q \to \infty$, has its two relative q-entropies positive (dashed curves). In view of the intimate relation of entropic inequalities with majorization [1, 22], we also analyse in figure 4 the probability that a state is completely majorized by both of its subsystems (solid line). It is shown in [22] that, if ρ_{AB} satisfies the reduction criterion, its two associated relative q-entropies are non-negative as well.

In the same work the authors assert that majorization is not implied by the relative entropy criteria. Our results confirm this assessment. In figure 4, the lower curves correspond to states ρ with $N = 2 \times N_2$, while the upper curves have $N = 3 \times N_2$. Majorization results and q-entropic criteria do coincide for two-qubits systems ($N_1 = N_2 = 2$). More generally, majorization probabilities are a lower bound to probabilities for relative q-entropic positivity, an interesting new result, as far as we know. Note also that the two approaches yield quite similar results in the $N = 3 \times N_2$ case.

4.4. Comparing more than two criteria together

We compare now the reduction criterion to the PPT one. The former is implied by the latter but is nonetheless a significant condition since its violation implies the possibility of recovering entanglement by distillation, which is as yet unclear for states that violate PPT [22]. Figure 5(a) depicts the probability that state ρ with $N = 3 \times N_2$ either

(1) has a positive partial transpose and does not violate the reduction criterion, or
(2) has a non-positive partial transpose and violates reduction.

Remember that in the case $N = 2 \times N_2$, the two criteria always coincide [1]. For $3 \times N_2$ the agreement between the two criteria becomes better and better as N_2 augments.

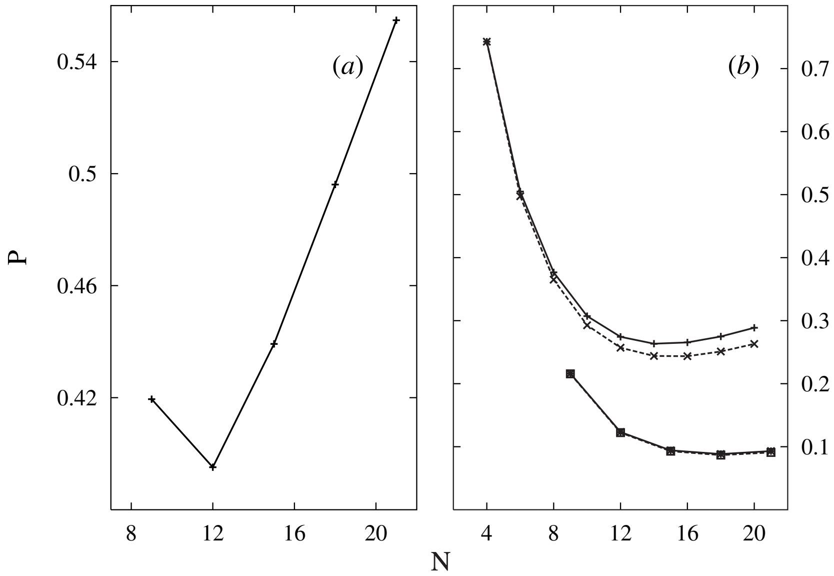

Figure 5. (*a*) Probability that the state ρ with $N = 3 \times N_2$ either has (i) a positive partial transpose *and* does not violate the reduction criterion, or (ii) has a non-positive partial transpose *and* violates reduction. In the case $N = 2 \times N_2$ the outcome is always unity. (*b*) Probability that (1) PPT and majorization (solid line) and, (2) PPT and the q-entropic criterion (dashed line) lead to the same conclusion regarding separability. Top: $N = 2 \times N_2$. Bottom: $N = 3 \times N_2$.

Of more interest is to compare the relations among PPT, majorization and the entropic criteria (figure 5(*b*)), since it is not yet known how the majorization criterion is related to other separability criteria such as PPT, undistillability and reduction [22]. In this vein, figure 5(*b*) plots the 'coincidence probability' between, respectively

(1) PPT and majorization (solid line), and
(2) PPT and the q-entropic criterion (dashed line).

The curves on the top correspond to $N = 2 \times N_2$, while those at the bottom to $N = 3 \times N_2$. In this last case the two curves agree with each other quite well.

The conclusion here is that, as N_2 augments, the probability of coincidence among the three criteria, and in particular between majorization and PPT (our main concern), rapidly diminishes at first, and stabilizes itself afterwards. For two qubits the three criteria do agree with each other to a large extent.

Figure 6(*a*) depicts the probability that, for a given state ρ,

(1) reduction and majorization (solid line) and
(2) reduction and the q-entropic criterion (dashed line)

yield the same conclusion as regards separability. Without PPT in the game, and opposite to what we encountered in figure 5, we find better coincidence for $N = 3 \times N_2$ systems (top) than for $N = 2 \times N_2$ (bottom). The deterioration of the degree of agreement as N_2 grows is similar to that of figure 5 though.

Figure 6(*b*) represents the probability that a state, for $q \to \infty$, either

(1) has both positive relative q-entropies and satisfies the majorization criterion, or
(2) has a negative relative q-entropy and is majorized by both of their subsystems.

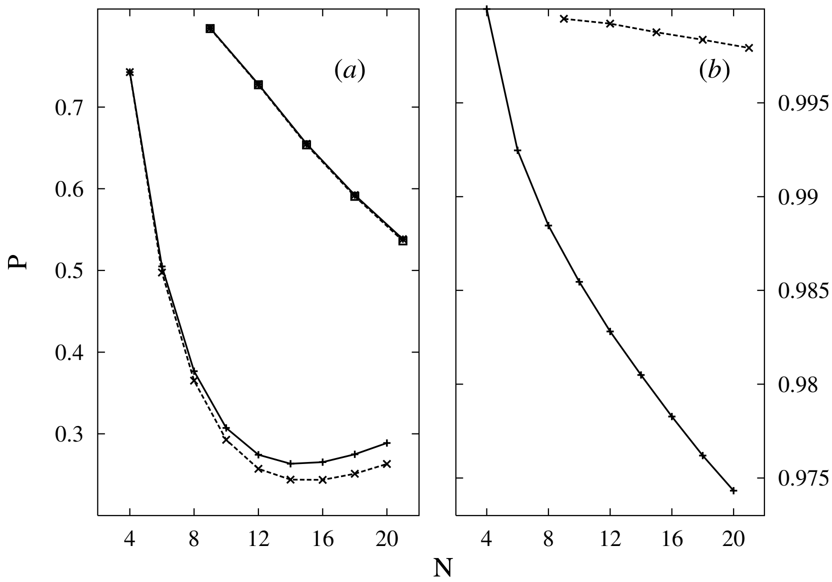

Figure 6. (a) Probability that reduction and majorization (solid line) and reduction and the q-entropic criterion (dashed line) yield the same conclusion regarding separability. Top: $N = 3 \times N_2$. Bottom: $N = 2 \times N_2$ (lower curves). (b) Probability that a state, for $q \to \infty$, either (i) has both positive relative q-entropies and fulfils majorization, or (ii) has a negative relative q-entropy and is majorized by both of their subsystems. The solid line corresponds to the case $N = 2 \times N_2$, while the dashed line corresponds to $N = 3 \times N_2$.

The solid line corresponds to the case $N = 2 \times N_2$, while the dashed line corresponds to the $N = 3 \times N_2$ instance. These results together with those of figures 4–5 could be read as implying that majorization and the q-entropic criteria provide almost the same answer for dimensions greater than or equal to $N = 3 \times N_2$.

Finally, in figure 7 we look for the probability P_{agree} that all criteria considered in the present work do lead to the same conclusion on the separability issue. P_{agree} is plotted as a function of the total dimension $N = N_1 \times N_2$, with $N_1 = 2$ (solid line) and $N_1 = 3$ (dashed line). The agreement is quite good for two qubits, deteriorates first as N_2 grows, and rapidly stabilizes itself around a value of 0.26 for $N_1 = 2$ and of 0.1 for $N_1 = 3$.

4.5. Distilling

Let us now consider the results plotted in figure 8. We ask first for the relative number of states that violate the reduction criterion and are thus distillable [37] (solid line), and appreciate the fact that, as N grows, so does the probability of finding distillable states. On the other hand, the probability of encountering states that violate the majorization criterion, represented by dashed lines, is much lower than that associated with distillation.

For both criteria, the upper solid line corresponds to the case $N = 2 \times N_2$, and the lower one to $N = 3 \times N_2$. The dashed curve with crosses represents the case $N = 2 \times N_2$, while that with squares indicates the $N = 3 \times N_2$ instance. The dependence on N_2 of the dashed curves (majorization violation) is not so strong as that of the solid ones (distillability). Our results are lower bounds to the total volume of states that can be distilled.

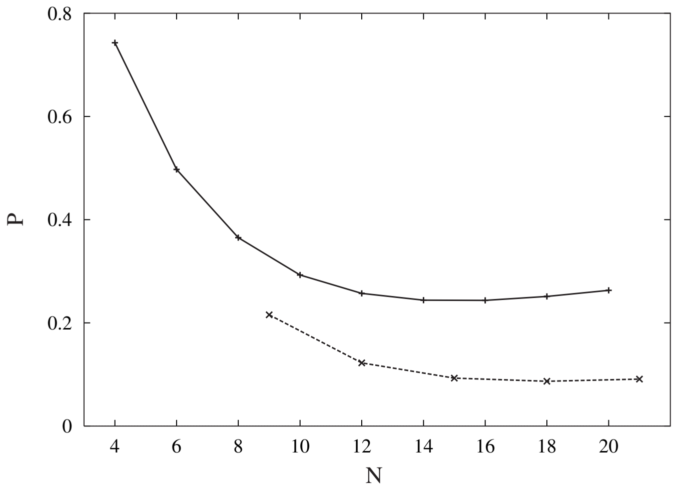

Figure 7. Total probability that all criteria considered in the present work lead to the same conclusion regarding separability. Probabilities are plotted as a function of the total dimension $N = N_1 \times N_2$, with $N_1 = 2$ (solid line) and $N_1 = 3$ (dashed line).

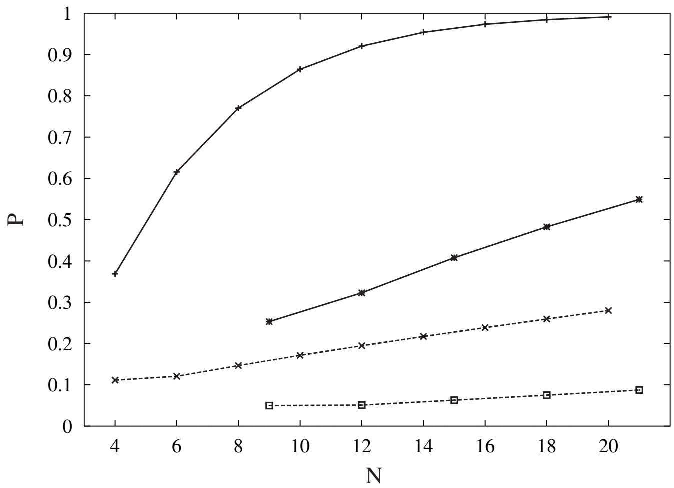

Figure 8. Solid line: probability that a state violates the reduction criterion. Dashed line: the same for violation of the majorization criterion. Top: $N = 2 \times N_2$. Bottom: $N = 3 \times N_2$. The dashed curve with crosses represents the case $N = 2 \times N_2$, while that with squares indicates the $N = 3 \times N_2$ instance.

5. Conclusions

We have performed a systematic numerical survey of the space of pure and mixed states of bipartite systems of dimension $2 \times N_2$ and $3 \times N_2$ in order to investigate the relationships ensuing among different separability criteria. Our main results are as follows:

- Regarding the line of separability implication, see our graph in figure 1 and *compare it with the similar one of Terhal's* (figure 3 of [1]). *Her question marks there*, referring to whether the states that satisfy both the reduction and the Peres–Horodecki criteria also verify the majorization strictures, *receive an answer from us in figure* 1.

- It is known that if a state satisfies PPT, it automatically verifies the reduction criterion [1]. In the present work, using the measure (6), we numerically prove that in the $N = 2 \times N_2$ instance, the converse is also true. In the $N = 3 \times N_2$ case, it is much more likely to encounter a state that verifies reduction than one that verifies PPT.
- We have numerically verified the assertion made in [22] that majorization is not implied by the relative entropic criteria. Majorization results and q-entropic criteria coincide for two-qubits systems. In general, majorization probabilities constitute lower bound for relative q-entropic positivity.
- Regarding the relation between majorization and PPT, the agreement between the criteria deteriorates as N_2 grows.
- For dimensions $\geqslant 3 \times N_2$, as illustrated by figures 4–5, majorization and the q-entropic criteria provide almost the same answers.

The present authors believe that the results of this numerical exploration, obtained on the basis of the measure (6), shed some light on the intricacies of the separability issue.

Acknowledgments

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Appendix. Q-information measures and the issue of quantum separability

There are several useful entropic (or information) measures for the investigation of a quite important subject: the violation of classical entropic inequalities by quantum entangled states. The von Neumann measure

$$S_1 = -\mathrm{Tr}\,(\rho \ln \rho) \tag{A1}$$

is important because of its relationship with the thermodynamic entropy. The q-entropy, which is a function of the quantity

$$\omega_q = \mathrm{Tr}\,(\rho^q) \tag{A2}$$

provides one with a whole family of entropic measures. In the limit $q \to 1$ these measures incorporate (A1) as a particular instance. Most of the applications of q-entropies to physics involve either the Rényi entropy [30],

$$S_q^{(R)} = \frac{1}{1-q} \ln(\omega_q) \tag{A3}$$

or the Tsallis entropy [31–33]

$$S_q^{(T)} = \frac{1}{q-1}(1-\omega_q). \tag{A4}$$

We reiterate that the von Neumann measure (A1) constitutes a particular instance of both Rényi's and Tsallis' entropies, obtained in the limit $q \to 1$. The most distinctive single property of Tsallis' entropy is its nonextensivity. The Tsallis entropy of a composite system $A \oplus B$ whose state is described by a factorizable density matrix, $\rho_{AB} = \rho_A \otimes \rho_B$, is given by Tsallis' q-additivity law,

$$S_q^{(T)}(\rho_{AB}) = S_q^{(T)}(\rho_A) + S_q^{(T)}(\rho_B) + (1-q) S_q^{(T)}(\rho_A) S_q^{(T)}(\rho_B). \tag{A5}$$

In contrast, Rényi's entropy is extensive. That is, if $\rho_{AB} = \rho_A \otimes \rho_B$,

$$S_q^{(R)}(\rho_{AB}) = S_q^{(R)}(\rho_A) + S_q^{(R)}(\rho_B). \tag{A6}$$

Tsallis' and Rényi's measures are related through

$$S_q^{(T)} = F\left(S_q^{(R)}\right) \tag{A7}$$

where the function F is given by

$$F(x) = \frac{1}{1-q}\{\mathrm{e}^{(1-q)x} - 1\}. \tag{A8}$$

An immediate consequence of equations (A7)–(A8) is that, for all non-vanishing values of q, Tsallis' measure $S_q^{(T)}$ is a monotonic increasing function of Rényi's measure $S_q^{(R)}$.

Considerably attention has been recently paid to a relative entropic measure based upon Tsallis' functional defined as

$$S_q^{(T)}(A|B) = \frac{S_q^{(T)}(\rho_{AB}) - S_q^{(T)}(\rho_B)}{1 + (1-q)S_q^{(T)}(\rho_B)}. \tag{A9}$$

Here ρ_{AB} designs an arbitrary quantum state of the composite system $A \oplus B$, not necessarily factorizable nor separable, and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. The relative q-entropy $S_q^{(T)}(B|A)$ is defined in a similar way as (A9), replacing ρ_B by $\rho_A = \mathrm{Tr}_B(\rho_{AB})$. The relative q-entropy (A9) has been recently studied in connection with the separability of density matrices describing composite quantum systems [27, 28]. For separable states, we have [22]

$$S_q^{(T)}(A|B) \geqslant 0 \qquad S_q^{(T)}(B|A) \geqslant 0. \tag{A10}$$

In contrast, there are entangled states that have negative relative q-entropies. That is, for some entangled states one (or both) of the inequalities (A10) are not verified.

Note that the denominator in (A9),

$$1 + (1-q)S_q^{(T)} = w_q > 0 \tag{A11}$$

is always positive. Consequently, as far as the sign of the relative entropy is concerned, the denominator in (A9) can be ignored. Besides, since Tsallis' entropy is a monotonic increasing function of Rényi's (see equations (A7)–(A8)), it is plain that (A9) has always the same sign as

$$S_q^{(R)}(A|B) = S_q^{(R)}(\rho_{AB}) - S_q^{(R)}(\rho_B). \tag{A12}$$

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